ISSN 0965-5425, Computational Mathematics and Mathematical Physics, 2009, Vol. 49, No. 11, pp. 1853–1868. © Pleiades Publishing, Ltd., 2009.
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First Moment Functions of the Solution to the Heat Equation with Random Coefficients

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Received January 26, 2009; in final form, April 28, 2009

Abstract—The expectation and the second moment of the solution to the linear inhomogeneous heat equation with random coefficients are found.

DOI: 10.1134/S0965542509110049

Key words: heat conduction, variational derivative, moment functions.

INTRODUCTION

An actual diffusion process depends on random factors. When the random coefficients are replaced by their expectations, the error introduced can be estimated if we know the expectation and variance of the diffusion process.

We find the expectation and the second moment of the solution to the Cauchy problem

$$\frac{\partial u(t,x)}{\partial t} = \varepsilon(t)\frac{\partial^2 u(t,x)}{\partial x_1^2} + \frac{\partial^2 u(t,x)}{\partial x_2^2} + \frac{\partial^2 u(t,x)}{\partial x_3^2} + f(t,x), \quad (1)$$

$$u(t_0, x) = u_0(x). \quad (2)$$

Here, $t \in [t_0, t_1] = T \subset \mathbb{R}$, $x \in \mathbb{R}^3$, u: $T \times \mathbb{R}^3 \longrightarrow \mathbb{R}$ is the unknown function, $\varepsilon(t) > 0$ and f: $T \times \mathbb{R}^3 \longrightarrow \mathbb{R}$ are stochastic processes, and $u_0 : \mathbb{R}^3 \longrightarrow \mathbb{R}$ is a stochastic process independent of ε or f.

The usual line of reasoning is as follows. Let ε, f, and u_0 be some realizations of the processes. Using a well-known formula, we can write the solution and find its expectation. The arising difficulties can be understood by considering a simple example. Consider the problem (here, $x \in \mathbb{R}$)

$$\frac{\partial u(t,x)}{\partial t} = \varepsilon(t)\frac{\partial^2 u(t,x)}{\partial x_1^2}, \quad u(0,x) = u_0(x),$$

where $\varepsilon > 0$ is a stochastic process defined by the distribution density $p_\varepsilon(t, \eta)$ and u_0 is a given function. The solution is

$$u_\varepsilon(t,x) = \frac{1}{2\sqrt{\pi\int_0^t \varepsilon(s)ds}} \int_{-\infty}^{+\infty} \exp\left[-\frac{(x-\tau)^2}{4\int_0^t \varepsilon(s)ds}\right] u_0(\tau)d\tau.$$

Let $\varepsilon > 0$ be a random variable with the distribution density $p_\varepsilon(\eta)$. Then

$$M(u_\varepsilon(t)) = \int_{-\infty}^{+\infty} \frac{1}{2\sqrt{\pi\eta t}} \int_{-\infty}^{+\infty} \exp\left[-\frac{(x-\tau)^2}{4\varepsilon t}\right] u_0(\tau)d\tau p_\varepsilon(\eta)d\eta.$$

If $\varepsilon(t)$ is a stochastic process defined by the distribution density $p_\varepsilon(t, \eta)$, then the expression

$$\mathbf{M}(u_\varepsilon(t)) = \int_{-\infty}^{+\infty} \frac{1}{2\sqrt{\pi\int_0^t \eta(s)ds}} \int_{-\infty}^{+\infty} \exp\left[-\frac{(x-\tau)^2}{4\int_0^t \eta(s)ds}\right] u_0(\tau)d\tau p_\varepsilon(t, \eta)d\eta$$

makes no sense, since the integral with respect to η is an operator rather than a function of η.

If a stochastic process is defined by a measure μ_ε on a space σ that is a realization of ε, then we have to evaluate the complicated path integral

$$M(u_\varepsilon(t)) = \int_\sigma \frac{1}{2\sqrt{\pi\int_0^t \varepsilon(s)ds}} \int_{-\infty}^{+\infty} \exp\left[-\frac{(x-\tau)^2}{4\int_0^t \varepsilon(s)ds}\right] u_0(\tau)d\tau d\mu_\varepsilon.$$

Other methods are developed for finding moment functions. Specifically, the Kolmogorov equations for the distribution density of the process $u(t, x)$ are used. Chains of equations for moment functions are constructed [1], which can sometimes be made closed and solved; asymptotic expansions in small random perturbations are constructed (see [2]).

There are other approaches available for finding moment functions [3, 4]. Formulas for the first two moment functions of the solution to a first-order scalar linear differential equation were derived in [3].

The approach used in this paper can be described as follows. We define (see Section 4) the auxiliary mapping

$$y(t, x, v, \omega) = M(u(t, x)e(v, \omega)), \tag{3}$$

where

$$e(v(\cdot), \omega(\cdot)) = \exp\left(i\int_T \varepsilon(s)v(s)ds + i\int_T\int_{R^3} f(s, \tau)\omega(s, \tau)dsd\tau\right),$$

$\mathbf{M}$ denotes the expectation with respect to the distribution function of ε and f. For y we obtain a deterministic problem with an explicit solution. Here, $M(u(t, x)) = y(t, x, 0, 0)$. Moreover, applying differentiation, we easily find the mixed moment functions

$$M(u(t, x)\varepsilon(s_1)\ldots\varepsilon(s_n)) = i^{-n}\frac{\delta^n y(t, x, 0, 0)}{\delta v(s_1)\ldots\delta v(s_n)}, \quad n = 1, 2, \ldots.$$

Let V be the Banach space of functions v: $T \longrightarrow \mathbb{R}$ equipped with the norm $\|v(\cdot)\|_V$ and W be the Banach space of functions $\omega : T \times \mathbb{R}^3 \longrightarrow \mathbb{R}$. Given numbers a and b, let $\chi(a, b, \cdot)$ denote the function defined according to the following rule: $\chi(a, b, s) = \operatorname{sgn}(s-a)$ if s lies in the interval with the endpoints at a and b and $\chi(a, b, s) = 0$ otherwise. For $a, b \in T$, assume that the functions $\chi(a, b, \cdot)$ belong to V and there is a constant $m > 0$ such that

$$\|v(\cdot)\|_V \le m\|v(\cdot)\|_{L_1} = m\int_T |v(t)|dt. \tag{4}$$

Let the sample functions of ε be such that $\int_T \varepsilon(t)v(t)dt$ is a linear bounded functional on V. Similarly, the realizations of f define the linear bounded functional $\int_T\int_{\mathbb{R}^3} f(t, x)\omega(t, x)dxdt$ on W. Let $\|x\|_3$ denote $(x_1^2 + x_2^2 + x_3^2)^{1/2}$ for $x \in \mathbb{R}^3$.

Assume that ε and f are defined by a characteristic functional; i.e.,

$$\varphi(v(\cdot), \omega(\cdot)) = Me(v(\cdot), \omega(\cdot)).$$

First, we study equations with a variational derivative.

1. FIRST-ORDER EQUATION WITH A VARIATIONAL DERIVATIVE

Let X be a Banach space of functions on the interval $T \subset \mathbb{R}$, $x: T \longrightarrow \mathbb{R}$, and $y: X \longrightarrow C$.

Definition 1 (see [5]). If the Fréchet differential $dy(x(\cdot), h)$ [6] of the functional y at a point $x_0(\cdot)$ has the form

$$dy(x_0(\cdot), h) = \int_T \varphi(t, x_0(\cdot))h(t)dt,$$

where the integral is understood in the sense of Lebesgue, then $\varphi : T \times X \longrightarrow C$ is called by the variational (functional) derivative of y at $x_0(\cdot)$ and is denoted by $\dfrac{\delta y(x_0(\cdot))}{\delta x(t)}$.

Consider the linear inhomogeneous problem

$$\frac{\partial y(t, x, v(\cdot))}{\partial t} = a_1(t)\frac{\delta y(t, x, v(\cdot))}{\delta v(t)} + a_2(t)y(t, x, v(\cdot)) + b(t, x, v(\cdot)), \tag{5}$$

$$y(t_0, x, v(\cdot)) = y_0(x, v(\cdot)) \tag{6}$$

for the unknown mapping $y : T \times \mathbb{R}^3 \times V \longrightarrow C$ in a neighborhood $\Omega \subset V$ of the point $v(\cdot) = 0$. The following result (see [5, p. 183]) is needed to prove Theorem 2.

Theorem 1. *If $a : T \longrightarrow C$ is a measurable bounded function on T, $y : V \longrightarrow C$ has a variational derivative $\delta y(a(\cdot)\chi(t_0, t, \cdot))/\delta v(s)$ that is summable with respect to s, and condition* (4) *is satisfied, then $f(t) = y(a(\cdot)\chi(t_0, t, \cdot))$ is differentiable almost everywhere on T and*

$$\frac{df(t)}{dt} = a(t)\frac{\delta y(a(\cdot)\chi(t_0, t, \cdot))}{\delta v(t)}.$$

Theorem 2. *Suppose that $a_i : T \longrightarrow C$, $i = 1, 2$, are measurable bounded functions on T; condition* (4) *is satisfied; $y_0(x, v(\cdot) + a_1(\cdot)\chi(t_0, t, \cdot))$ has a summable variational derivative in Ω; $b : T \times \mathbb{R}^3 \times V \longrightarrow C$ is summable on T with respect to the first variable; and there exists a τ-summable variational derivative $\delta b(s, x, v(\cdot) + a_1(\cdot)\chi(s, t, \cdot))/\delta v(\tau)$ that has a summable majorant on T for $v(\cdot) \in \Omega$: $|\delta b(s, x, v(\cdot) + a_1(\cdot)\chi(t_0, t, \cdot))/\delta v(\tau)| \le m(s)$. Then*

$$\begin{aligned} y(t, x, v(\cdot)) &= \exp\left(\int_{t_0}^{t} a_2(s)ds\right)y_0(x, v(\cdot) + a_1(\cdot)\chi(t_0, t, \cdot)) \\ &+ \int_{t_0}^{t}\exp\left(\int_{s}^{t} a_2(\tau)d\tau\right)b(s, x, v(\cdot) + a_1(\cdot)\chi(s, t, \cdot))ds \end{aligned} \tag{7}$$

is a solution to problem (5), (6) *in Ω.*

Proof. Since $b(s, x, v(\cdot) + a_1(\cdot)\chi(t_0, t, \cdot))$ has a variational derivative with respect to v in Ω, it is continuous with respect to v in Ω. Furthermore, since b is summable with respect to the first variable, $b(s, x, v(\cdot) + a_1(\cdot)\chi(s, t, \cdot))$ is summable with respect to s on T. Due to the summable majorant $m(s)$, the variational derivative of the integral in (7) can be calculated by variational differentiation under the integral sign. Applying Theorem 1, we find

$$\frac{\partial y(t, x, v(\cdot))}{\partial t} = a_2(t)\exp\left(\int_{t_0}^{t} a_2(s)ds\right) y_0(x, v(\cdot) + a_1(\cdot)\chi(t_0, t, \cdot))$$

$$+ \exp\left(\int_{t_0}^{t} a_2(s)ds\right) a_1(t)\frac{\delta}{\delta v(t)} y_0(x, v(\cdot) + a_1(\cdot)\chi(t_0, t, \cdot)) + b(t, x, v(\cdot))$$

$$+ \int_{t_0}^{t} a_2(t)\exp\left(\int_{s}^{t} a_2(\tau)d\tau\right) b(s, x, v(\cdot) + a_1(\cdot)\chi(s, t, \cdot))ds$$

$$+ \int_{t_0}^{t} \exp\left(\int_{s}^{t} a_2(\tau)d\tau\right) a_1(t)\frac{\delta b(s, x, v(\cdot) + a_1(\cdot)\chi(s, t, \cdot))}{\delta v(t)} ds.$$

The variational derivative is calculated as

$$\frac{\delta y(t, x, v(\cdot))}{\delta v(t)} = \exp\left(\int_{t_0}^{t} a_2(s)ds\right) \frac{\delta}{\delta v(t)} y_0(x, v(\cdot) + a_1(\cdot)\chi(t_0, t, \cdot))$$

$$+ \int_{t_0}^{t} \exp\left(\int_{s}^{t} a_2(\tau)d\tau\right) \frac{\delta b(s, x, v(\cdot) + a_1(\cdot)\chi(s, t, \cdot))}{\delta v(t)} ds.$$

Substituting this into (5) gives a correct equality for $v(\cdot) \in \Omega$ for almost all $t \in T$. The theorem is proved.

2. THIRD-ORDER EQUATION WITH A VARIATIONAL DERIVATIVE

Let $x \in \mathbb{R}^3$; $f: \mathbb{R}^3 \times \mathbb{R}^m \longrightarrow C$; ξ be a vector with coordinates ξ_1, ξ_2, and ξ_3; $F_x[f(x, y)](\xi)$ denote the Fourier transform with respect to x; similar notation be used for the inverse Fourier transform; and $\overset{x}{*}$ denote the convolution with respect to x (see [7]).

Consider the Cauchy problem for the third-order differential equation

$$\frac{\partial y(t, x, v(\cdot))}{\partial t} = -i\frac{\delta}{\delta v(t)}\frac{\partial^2}{\partial x_1^2} y(t, x, v(\cdot)) + \frac{\partial^2}{\partial x_2^2} y(t, x, v(\cdot)) + \frac{\partial^2}{\partial x_3^2} y(t, x, v(\cdot)) + b(t, x, v(\cdot)), \quad (8)$$

$$y(t_0, x, v(\cdot)) = y_0(x, v(\cdot)). \quad (9)$$

Here, $t \in T \subset \mathbb{R}^3$, $x \in \mathbb{R}^3$, $b: T \times \mathbb{R}^3 \times V \longrightarrow C$ and $y_0: \mathbb{R}^3 \times V \longrightarrow C$ are given mappings, and y is the unknown mapping. In the theorem below, the arguments are omitted from y_0 and b (namely, x and $v(\cdot) + i\xi_1^2\chi(t_0, t, \cdot)$ from y_0 and $s, x, v(\cdot) + i\xi_1^2\chi(s, t, \cdot)$ from b) and $\delta(x)$ denotes the delta function.

Theorem 3. *Suppose that condition* (4) *is satisfied and there exists a neighborhood* Ω *of the point* $v(\cdot) = 0$ *such that, for all* $v(\cdot) \in \Omega$, *the functions*

$$|y_0|, \quad \left|\frac{\delta y_0}{\delta v(t)}\right|, \quad \left|F_{x_1}\left[\frac{\delta y_0}{\delta v(t)}\right](\xi)\right|, \quad |\xi_1 F_{x_1}[y_0](\xi)|, \quad \xi_1^2|F_{x_1}[y_0](\xi)|, \quad \left|\xi_1 F_{x_1}\left[\frac{\delta y_0}{\delta v(t)}\right](\xi)\right|,$$

$$\xi_1^2\left|F_{x_1}\left[\frac{\delta y_0}{\delta v(t)}\right](\xi)\right|, \quad |b|, \quad \left|\frac{\delta b}{\delta v(t)}\right|, \quad \left|F_{x_1}\left[\frac{\delta b}{\delta v(t)}\right](\xi)\right|, \quad |\xi_1||F_{x_1}[b](\xi)|, \quad \xi_1^2|F_{x_1}[b](\xi)|,$$

$$|\xi_1|\left|F_{x_1}\left[\frac{\delta b}{\delta v(t)}\right](\xi)\right|, \quad \xi_1^2\left|F_{x_1}\left[\frac{\delta b}{\delta v(t)}\right](\xi)\right|$$

are bounded for $t \in T$ *and* $s \in T$ *by summable functions on* $\mathbb{R}^3$. *Then the solution to problem* (8), (9) *is given by the formula*

$$y(t, x, v(\cdot)) = \frac{1}{4\pi(t-t_0)}\exp\left[-\frac{x_2^2+x_3^2}{4(t-t_0)}\right]\delta(x_1) \overset{x}{*} F_{\xi_1}^{-1}[F_{x_1}[y_0(x, v(\cdot)+i\xi_1^2\chi(t_0, t, \cdot))](\xi_1)](x_1)$$
$$+\int_{t_0}^{t}\frac{1}{4\pi(t-s)}\exp\left[-\frac{x_2^2+x_3^2}{4(t-s)}\right]\delta(x_1) \overset{x}{*} F_{\xi_1}^{-1}[F_{x_1}[b(s, x, v(\cdot)+i\xi_1^2\chi(s, t, \cdot))](\xi_1)](x_1)ds. \quad (10)$$

Proof. Assume that a Fourier transform with respect to x exists for the solution to problem (8), (9). Applying the Fourier transform to (8) and (9) yields

$$\frac{\partial}{\partial t}F_x[y(t, x, v(\cdot))](\xi) = -i\frac{\delta}{\delta v(t)}(-i\xi_1)^2 F_x[y(t, x, v(\cdot))](\xi)$$
$$+(-i\xi_2)^2F_x[y(t, x, v(\cdot))](\xi)+(-i\xi_3)^2F_x[y(t, x, v(\cdot))](\xi)+F_x[b(t, x, v(\cdot))](\xi),$$
$$F_x[y(t_0, x, v(\cdot))](\xi) = F_x[y_0(x, v(\cdot))](\xi).$$

This problem has the form of (5), (6) and ξ is a parameter. Using formula (7),

$$F_x[y(t, x, v(\cdot))](\xi) = \exp[-(\xi_2^2+\xi_3^2)(t-t_0)]F_x[y_0(x, v(\cdot)+i\xi_1^2\chi(t_0, t, \cdot))](\xi)$$
$$+\int_{t_0}^{t}\exp[-(\xi_2^2+\xi_3^2)(t-s)]F_x[b(s, x, v(\cdot)+i\xi_1^2\chi(s, t, \cdot))](\xi)ds.$$

Applying the inverse Fourier transform, we obtain

$$y(t, x, v(\cdot)) = F_\xi^{-1}\{\exp[-(\xi_2^2+\xi_3^2)(t-t_0)]\}(x) \overset{x}{*} F_\xi^{-1}\{F_x[y_0(x, v(\cdot)+i\xi_1^2\chi(t_0, t, \cdot))](\xi)\}(x)$$
$$+\int_{t_0}^{t}F_\xi^{-1}\{\exp[-(\xi_2^2+\xi_3^2)(t-s)]\}(x) \overset{x}{*} F_\xi^{-1}\{F_x[b(s, x, v(\cdot)+i\xi_1^2\chi(s, t, \cdot))](\xi)\}(x)ds$$
$$= \frac{1}{4\pi(t-t_0)}\exp\left[-\frac{x_2^2+x_3^2}{4(t-t_0)}\right]\delta(x_1) \overset{x}{*} F_\xi^{-1}\{F_x[y_0(x, v(\cdot)+i\xi_1^2\chi(t_0, t, \cdot))](\xi)\}(x)$$
$$+\int_{t_0}^{t}\frac{1}{4\pi(t-s)}\exp\left[-\frac{x_2^2+x_3^2}{4(t-s)}\right]\delta(x_1) \overset{x}{*} F_\xi^{-1}\{F_x[b(s, x, v(\cdot)+i\xi_1^2\chi(s, t, \cdot))](\xi)\}(x)ds.$$

The theorem assumptions on the existence of summable majorants ensure differentiability under the integral sign with respect to the required variables. Using the properties of Fourier transforms gives $F_\xi^{-1}\{F_x[y_0(x, v(\cdot)+i\xi_1^2\chi(t_0, t, \cdot))](\xi)\}(x) = F_{\xi_1}^{-1}\{F_{\xi_2\xi_3}^{-1}[F_{x_2x_3}[F_{x_1}[y_0(x, v(\cdot) + i\xi_1^2\chi(t_0, t, \cdot))](\xi_1)](\xi_2\xi_3)](x_2x_3)\}(x_1) = F_{\xi_1}^{-1}\{F_{x_1}[y_0(x, v(\cdot) + i\xi_1^2\chi(t_0, t, \cdot))](\xi_1)\}(x_1)$. In a similar manner, we obtain $F_\xi^{-1}\{F_x[b(s, x, v(\cdot)+i\xi_1^2\chi(s, t, \cdot))](\xi)\}(x) = F_{\xi_1}^{-1}\{F_{\xi_2\xi_3}^{-1}[F_{x_2x_3}[F_{x_1}[b(s, x, v(\cdot) + i\xi_1^2\chi(s, t, \cdot))](\xi_1)](\xi_2\xi_3)](x_2x_3)\}(x_1) = F_{\xi_1}^{-1}\{F_{x_1}[b(s, x, v(\cdot)+i\xi_1^2\chi(s, t, \cdot))](\xi_1)\}(x_1)$. Plugging these expressions into the previous equality yields formula (10).

The theorem is proved.

3. EXPECTATION OF THE SOLUTION TO THE HEAT EQUATION

We introduce the notation

$$y(t, x, v(\cdot), \omega(\cdot)) = M(u(t, x)e(v(\cdot), \omega(\cdot))),$$

where the expectation is calculated with respect to the distribution function of u_0, ε, and f in problem (1), (2).

Multiplying (1) and (2) by $e(v(\cdot), \omega(\cdot))$, we find the expectation with respect to the distribution function of u_0, ε, and f. If the corresponding derivatives of y exist, then the last equalities are written as

$$\frac{\partial y(t, x, v(\cdot), \omega(\cdot))}{\partial t} = -i\frac{\delta}{\delta v(t)}\frac{\partial^2}{\partial x_1^2}y(t, x, v(\cdot), \omega(\cdot)) + \frac{\partial^2}{\partial x_2^2}y(t, x, v(\cdot), \omega(\cdot)) + \frac{\partial^2}{\partial x_3^2}y(t, x, v(\cdot), \omega(\cdot)) - i\frac{\delta\varphi(v(\cdot), \omega(\cdot))}{\delta\omega(t, x)}, \tag{11}$$

$$y(t_0, x, v(\cdot), \omega(\cdot)) = M(u_0(x))\varphi(v(\cdot), \omega(\cdot)), \tag{12}$$

where φ is the characteristic functional of ε and f. Here, we used the independence of u_0 from ε and f.

Problem (11), (12) is deterministic, but Eq. (11) is not conventional, since it involves variational differentiation. Naturally, the form of y prompts the following definition.

Definition 2. The *expectation* of the solution to problem (1), (2) is

$$\mathrm{Mu}(t, x) = y(t, x, 0, 0), \tag{13}$$

where y solves problem (11), (12) in a neighborhood of (0, 0) in $V \times W$. If y solves problem (11), (12) in the sense of generalized functions (distributions), then (13) is called the *generalized expectation* of the solution to problem (1), (2).

If we set $M(u_0(x))\varphi(v(\cdot), \omega(\cdot)) = y_0(x, v(\cdot), \omega(\cdot))$ and $-i\delta\varphi(v(\cdot), \omega(\cdot))/\delta\omega(t, x) = b(t, x, v(\cdot), \omega(\cdot))$, then problem (11), (12) has the form of (8), (9) for every fixed $\omega(\cdot)$.

Theorem 4. *Let* $M(u_0(x))$ *be summable on* $\mathbb{R}^3$, *and let the conditions of Theorem* 3 *be satisfied for every* $\omega(\cdot)$ *from a neighborhood of the origin in* W. *Then the solution to problem* (11), (12) *is given by the formula*

$$y(t, x, v(\cdot), \omega(\cdot)) = \frac{1}{4\pi(t-t_0)}\exp\left[-\frac{x_2^2+x_3^2}{4(t-t_0)}\right] \overset{x_2x_3}{*} \{M(u_0(x)) \overset{x_1}{*} F_{\xi_1}^{-1}[\varphi(v(\cdot) + i\xi_1^2\chi(t_0, t, \cdot), \omega(\cdot))](x_1)\} - i\int_{t_0}^{t}\frac{1}{4\pi(t-s)}\exp\left[-\frac{x_2^2+x_3^2}{4(t-s)}\right] \overset{x_2x_3}{*} F_{\xi_1}^{-1}\left\{F_{x_1}\left[\frac{\delta\varphi(v(\cdot) + i\xi_1^2\chi(s, t, \cdot), \omega(\cdot))}{\delta\omega(s, x)}\right](\xi_1)\right\}(x_1)ds. \tag{14}$$

Proof. Using formula (10), we find a solution to problem (11), (12):

$$y = \frac{1}{4\pi(t-t_0)}\exp\left(-\frac{x_2^2+x_3^2}{4(t-t_0)}\right)\delta(x_1) \overset{x}{*} F_{\xi_1}^{-1}[F_{x_1}[M(u_0(x))\varphi(v(\cdot) + i\xi_1^2\chi(t_0, t, \cdot), \omega(\cdot))](\xi_1)](x_1) - i\int_{t_0}^{t}\frac{1}{4\pi(t-s)}\exp\left[-\frac{x_2^2+x_3^2}{4(t-s)}\right]\delta(x_1) \overset{x}{*} F_{\xi_1}^{-1}\left\{F_{x_1}\left[\frac{\delta\varphi(v(\cdot) + i\xi_1^2\chi(s, t, \cdot), \omega(\cdot))}{\delta\omega(s, x)}\right](\xi_1)\right\}(x_1)ds.$$

Since the inverse Fourier transform of the product of functions is transformed into the convolution of the inverse Fourier transforms of the multipliers, the last relation yields (14). The theorem is proved.

Theorem 5. *Under the conditions of Theorem* 3, *the expectation of the solution to problem* (1), (2) *is given by the formula*

$$Mu(t, x) = \frac{1}{4\pi(t-t_0)}\exp\left[-\frac{x_2^2+x_3^2}{4(t-t_0)}\right] \overset{x_2x_3}{*} [M(u_0(x)) \overset{x_1}{*} F_{\xi_1}^{-1}(\varphi(i\xi_1^2\chi(t_0, t, \cdot), 0))(x_1)] - i\int_{t_0}^{t}\frac{1}{4\pi(t-s)}\exp\left[-\frac{x_2^2+x_3^2}{4(t-s)}\right] \overset{x_2x_3}{*} F_{\xi_1}^{-1}\left[F_{x_1}\left(\frac{\delta\varphi(i\xi_1^2\chi(s, t, \cdot), 0)}{\delta\omega(s, x)}\right)(\xi_1)\right](x_1)ds. \tag{15}$$

The proof follows from (13) and (14).

4. SPECIAL CASES

Formula (15) is rather general. It was derived even without assuming the independence of ε and f.

4.1. The case of independent ε and f. The characteristic functional $\varphi(v(\cdot), \omega(\cdot))$ is then the product of the characteristic functionals $\varphi_\varepsilon(v(\cdot))$ and $\varphi_f(\omega(\cdot))$ defining ε and f.

Theorem 6. *Suppose that the stochastic processes u_0, ε, and f in problem* (1), (2) *are independent; condition* (4) *is satisfied; the characteristic functional $\varphi_\varepsilon : V \longrightarrow C$ of the process ε has a variational derivative with respect to $v \in L_1(T)$; and the functions $M(u_0(x))$ and $M(f(t, x))$ are locally summable. Then the generalized expectation of the solution to problem* (1), (2) *is given by the formula*

$$Mu(t,x) = \frac{1}{4\pi(t-t_0)}\exp\left[-\frac{x_2^2+x_3^2}{4(t-t_0)}\right] \overset{x_2x_3}{*} \{M(u_0(x)) \overset{x_1}{*} F_{\xi_1}^{-1}[\varphi_\varepsilon(i\xi_1^2\chi(t_0,t,\cdot))](x_1)\}$$
$$+ \int_{t_0}^{t}\frac{1}{4\pi(t-s)}\exp\left[-\frac{x_2^2+x_3^2}{4(t-s)}\right] \overset{x_2x_3}{*} \{F_{\xi_1}^{-1}[\varphi_\varepsilon(i\xi_1^2\chi(s,t,\cdot))](x_1) \overset{x_1}{*} Mf(s,x)\}ds. \quad (16)$$

Proof. Note that $\dfrac{\delta\varphi_f(0)}{\delta\omega(t,x)} = iMf(t,x)$, $\varphi_f(0) = 1$, and $\varphi(v(\cdot), \omega(\cdot)) = \varphi_\varepsilon(v(\cdot))\varphi_f(\omega(\cdot))$. Using formulas (13) and (14), we find

$$Mu(t,x) = \frac{1}{4\pi(t-t_0)}\exp\left[-\frac{x_2^2+x_3^2}{4(t-t_0)}\right] \overset{x_2x_3}{*} \{M(u_0(x)) \overset{x_1}{*} F_{\xi_1}^{-1}[\varphi_\varepsilon(i\xi_1^2\chi(t_0,t,\cdot))\varphi_f(0)](x_1)\}$$
$$- i\int_{t_0}^{t}\frac{1}{4\pi(t-s)}\exp\left[-\frac{x_2^2+x_3^2}{4(t-s)}\right] \overset{x_2x_3}{*} F_{\xi_1}^{-1}\left\{F_{x_1}\left[\varphi_\varepsilon(i\xi_1^2\chi(s,t,\cdot))\frac{\delta\varphi_f(0)}{\delta\omega(s,x)}\right](\xi_1)\right\}(x_1)ds$$
$$= \frac{1}{4\pi(t-t_0)}\exp\left[-\frac{x_2^2+x_3^2}{4(t-t_0)}\right] \overset{x_2x_3}{*} \{M(u_0(x)) \overset{x_1}{*} F_{\xi_1}^{-1}[\varphi_\varepsilon(i\xi_1^2\chi(t_0,t,\cdot))](x_1)\}$$
$$+ \int_{t_0}^{t}\frac{1}{4\pi(t-s)}\exp\left[-\frac{x_2^2+x_3^2}{4(t-s)}\right] \overset{x_2x_3}{*} F_{\xi_1}^{-1}[\varphi_\varepsilon(i\xi_1^2\chi(s,t,\cdot))F_{x_1}[Mf(s,x)](\xi_1)](x_1)ds.$$

The inverse Fourier transform under the integral sign is expressed in terms of the convolution, which yields (16). The theorem is proved.

4.2. The Case of a Gaussian process ε. The Gaussian stochastic process is defined by the characteristic functional

$$\varphi_\varepsilon(v(\cdot)) = \exp\left(i\int_T M\varepsilon(s)v(s)ds - \frac{1}{2}\int_T\int_T b(s_1,s_2)v(s_1)v(s_2)ds_1ds_2\right),$$

where $M\varepsilon(s)$ is the expectation of ε and $b(s_1, s_2) = M(\varepsilon(s_1)\varepsilon(s_2)) - M\varepsilon(s_1)M\varepsilon(s_2)$ is the covariance function of ε.

Theorem 7. *Let ε be a Gaussian stochastic process independents of f, $M\varepsilon(t) > 0$, $M\varepsilon(\cdot) \in L_\infty(T)$, b: $T \times T \longrightarrow \mathbb{R}$ be a measurable bounded function, and $M(u_0(\cdot))$ and $M(f(t, \cdot))$ be locally summable functions. Then the generalized expectation of the solution to problem* (1), (2) *is calculated by the formula*

$$Mu(t,x) = \frac{1}{4\pi(t-t_0)}\exp\left[-\frac{x_2^2+x_3^2}{4(t-t_0)}\right] \overset{x_2x_3}{*} \left[\frac{1}{2\sqrt{\pi\int_{t_0}^{t}M\varepsilon(s)ds}}\exp\left(-\frac{x_1^2}{4\int_{t_0}^{t}M\varepsilon(s)ds}\right)\right] \overset{x_1}{*} \sum_{k=0}^{\infty}\frac{1}{2^k k!}\left(\int_{t_0}^{t}\int_{t_0}^{t}b(s_1,s_2)ds_1ds_2\right)^k$$

$$\times\frac{\partial^{4k}}{\partial x_1^{4k}}M(u_0(x)) + \int_{t_0}^{t}\frac{1}{4\pi(t-s)}\exp\left(-\frac{x_2^2+x_3^2}{4(t-s)}\right) \overset{x_2x_3}{*} \left[\frac{1}{2\sqrt{\pi\int_{t_0}^{t}M\varepsilon(\tau)ds}}\exp\left(-\frac{x_1^2}{4\int_{t_0}^{t}M\varepsilon(\tau)ds}\right)\right] \overset{x_1}{*} \sum_{k=0}^{\infty}\frac{1}{2^k k!} \tag{17}$$

$$\times\left(\int_{s}^{t}\int_{s}^{t}b(s_1,s_2)ds_1ds_2\right)^k\frac{\partial^{4k}}{\partial x_1^{4k}}Mf(s,x)ds.$$

Proof. The functional φ_ε has a variational derivative, and expression (16) is defined in the generalized sense. It is well known (see [7]) that

$$F_{\xi}^{-1}\left[\exp\left(-\xi_1^2\int_{t_0}^{t}M\varepsilon(s)ds\right)\right](x) = \left(4\pi\int_{t_0}^{t}M\varepsilon(s)ds\right)^{-1/2}\exp\left(-\frac{x_1^2}{4\int_{t_0}^{t}M\varepsilon(s)ds}\right) \quad \text{for} \quad \int_{t_0}^{t}M\varepsilon(s)ds > 0.$$

The last condition holds since $M\varepsilon(s) > 0$. Furthermore,

$$F_{\xi_1}^{-1}[\varphi_\varepsilon(i\xi_1^2\chi(t_0,t,\cdot))](x_1) = F_{\xi_1}^{-1}\left[\exp\left(-\xi_1^2\int_{t_0}^{t}M\varepsilon(s)ds\right)\right](x_1) \overset{x}{*} F_{\xi_1}^{-1}\left[\exp\left(\frac{1}{2}\xi_1^4\int_{t_0}^{t}\int_{t_0}^{t}b(s_1,s_2)ds_1ds_2\right)\right](x_1).$$

Define $B(t_0,t) = \int_{t_0}^{t}\int_{t_0}^{t}b(s_1,s_2)ds_1ds_2$. Then

$$F_{\xi_1}^{-1}\left[\exp\left(\frac{1}{2}\xi_1^4B(t_0,t)\right)\right](x_1) = \sum_{k=0}^{\infty}\frac{B^k(t_0,t)}{2^k k!}F_{\xi_1}^{-1}[\xi_1^4](x_1) = \sum_{k=0}^{\infty}\frac{B^k(t_0,t)}{2^k k!}\frac{d^{4k}}{dx_1^{4k}}\delta(x_1).$$

Moreover,

$$F_{\xi_1}^{-1}\left[\exp\left(\frac{1}{2}B(s,t)\xi_1^4\right)\right](x_1) \overset{x_1}{*} Mf(s,x) = \int_R\left(\sum_{k=0}^{\infty}\frac{B^k(s,t)}{2^k k!}\frac{\partial^{4k}}{\partial x_1^{4k}}\delta(x-\eta)Mf(s,\eta)\right)d\eta$$

$$= \sum_{k=0}^{\infty}\frac{B^k(t,s)}{2^k k!}\frac{\partial^{4k}}{\partial x_1^{4k}}Mf(s,x).$$

Substituting these relations into (16) gives (17). The theorem is proved.

5. AUXILIARY CAUCHY PROBLEM

The second moment of the solution to problem (1), (2) is found in the same manner as the expectation. Define the auxiliary mapping

$$z(t,t_1,x,\tau,v(\cdot),\omega(\cdot)) = M(u(t,x)u(t_1,\tau)e(v(\cdot),\omega(\cdot))).$$

Equation (1) is multiplied by $u(t_1, \tau)e(v(\cdot), \omega(\cdot))$ and is averaged over the distribution function of ε, f, and u_0. Formally, this equality is written in terms of z and y as

$$\begin{aligned}\frac{\partial z(t, t_1, x, \tau, v(\cdot), \omega(\cdot))}{\partial t} &= -i\frac{\delta}{\delta v(t)}\frac{\partial^2}{\partial x_1^2}z(t, t_1, x, \tau, v(\cdot), \omega(\cdot)) + \frac{\partial^2}{\partial x_2^2}z(t, t_1, x, \tau, v(\cdot), \omega(\cdot)) \\ &+ \frac{\partial^2}{\partial x_3^2}z(t, t_1, x, \tau, v(\cdot), \omega(\cdot)) - i\frac{\delta y(t_1, \tau, v(\cdot), \omega(\cdot))}{\delta\omega(t, x)}.\end{aligned} \tag{18}$$

Unfortunately, the initial value $z(t_0, t_1, x, \tau, v(\cdot), \omega(\cdot))$ cannot be obtained from condition (2). However, multiplying (2) by $u(t_0, \tau)e(v(\cdot), \omega(\cdot))$ and averaging over the distribution function of ε, f, and u_0 yields

$$z(t_0, t_0, x, \tau, v(\cdot), \omega(\cdot)) = M(u_0(x)u_0(\tau))\varphi(v(\cdot), \omega(\cdot)). \tag{19}$$

By definition, z is symmetric about (t, x) and (t_1, τ); i.e., $z(t, t_1, x, \tau, v(\cdot), \omega(\cdot)) = z(t_1, t, \tau, x, v(\cdot), \omega(\cdot))$. Using this feature, we can find z.

Theorem 8. *Suppose that condition* (4) *is satisfied*; $M(u_0(x))$, $M(u_0(\tau))$, *and* $M(u_0(x)u_0(\tau))$ *are locally summable in an neighborhood of the point* $(0, 0) \in V \times W$; *and there exist variational derivatives*

$$\frac{\delta\varphi(v(\cdot) + i\eta_1^2\chi(s, t_1, \cdot) + i\xi_1^2\chi(t_0, t, \cdot), \omega(\cdot))}{\delta\omega(s, \tau)}, \quad \frac{\delta\varphi(v(\cdot) + i\xi_1^2\chi(s, t, \cdot) + i\eta_1^2\chi(t_0, t_1, \cdot), \omega(\cdot))}{\delta\omega(s, x)},$$

$$\frac{\delta^2\varphi(v(\cdot) + i\eta_1^2\chi(\theta, t_1, \cdot) + i\xi_1^2\chi(s, t, \cdot), \omega(\cdot))}{\delta\omega(s, x)\delta\omega(\theta, \tau)}.$$

Then the (t, x)- and (t_1, τ)-symmetric generalized solution to problem (18), (19) is given by the formula

$$\begin{aligned} z &= \frac{1}{4\pi(t-t_0)}\exp\left[-\frac{x_2^2+x_3^2}{4(t-t_0)}\right] \overset{x_2x_3}{*} \Bigg\{\frac{1}{4\pi(t_1-t_0)} \\ &\times \exp\left[-\frac{\tau_2^2+\tau_3^2}{4(t_1-t_0)}\right] \overset{\tau_2\tau_3}{*} \{M(u_0(x)u_0(\tau)) \overset{\tau_1}{*} F_{\xi_1}^{-1}[F_{\eta_1}^{-1}[\varphi(v(\cdot) + i\eta_1^2\chi(t_0, t_1, \cdot) + i\xi_1^2\chi(t_0, t, \cdot), \omega(\cdot))](\tau_1)](x_1)\} \\ &- i\frac{1}{4\pi(t-t_0)}\exp\left(-\frac{x_2^2+x_3^2}{4(t-t_0)}\right) \overset{x_2x_3}{*} \Bigg\{M(u_0(x)) \overset{x_1}{*} \int_{t_0}^{t_1}\frac{1}{4\pi(t_1-s)} \\ &\times \exp\left(-\frac{\tau_2^2+\tau_3^2}{4(t_1-s)}\right) \overset{\tau_2\tau_3}{*} F_{\xi_1}^{-1}\left[F_{\eta_1}^{-1}\left[F_{x_1}\left[\frac{\delta}{\delta\omega(s, \tau)}\varphi(v(\cdot) + i\eta_1^2\chi(s, t_1, \cdot) + i\xi_1^2\chi(t_0, t, \cdot), \omega(\cdot))\right](\eta_1)\right](\tau_1)\right](x_1)ds\Bigg\} \\ &- i\frac{1}{4\pi(t_1-t_0)}\exp\left(-\frac{\tau_2^2+\tau_3^2}{4(t_1-t_0)}\right) \overset{\tau_2\tau_3}{*} \Bigg\{M(u_0(\tau)) \overset{\tau_1}{*} \int_{t_0}^{t}\frac{1}{4\pi(t-s)}\exp\left(-\frac{x_2^2+x_3^2}{4(t-s)}\right) \overset{x_2x_3}{*} F_{\eta_1}^{-1}\left[F_{\xi_1}^{-1}\left[F_{x_1}\left[\frac{\delta}{\delta\omega(s, x)}\varphi(v(\cdot) \right.\right.\right. \\ &\left.\left.\left. + i\eta_1^2\chi(t_0, t_1, \cdot) + i\xi_1^2\chi(s, t, \cdot), \omega(\cdot))\right](\xi_1)\right](x_1)\right](\tau_1)ds\Bigg\} \\ &- \int_{t_0}^{t}ds\int_{t_0}^{t_1}\frac{1}{4\pi(t-s)}\exp\left(-\frac{x_2^2+x_3^2}{4(t-s)}\right) \overset{x_2x_3}{*} \Bigg\{\frac{1}{4\pi(t_1-\theta)}\exp\left[-\frac{\tau_2^2+\tau_3^2}{4(t_1-\theta)}\right] \overset{\tau_2\tau_3}{*} F_{\xi_1}^{-1}\Big[F_{x_1}\Big[F_{\eta_1}^{-1}\Big[F_{x_1} \end{aligned} \tag{20}$$

$$\times\left[\frac{\delta^2\varphi(v(\cdot)+i\eta_1^2\chi(\theta,t_1,\cdot)+i\xi_1^2\chi(s,t,\cdot),\omega(\cdot))}{\delta\omega(s,x)\delta\omega(\theta,\tau)}\right](\eta_1)\Big](\tau_1)\Big](\xi_1)\Big](x_1)\Big\}d\theta.$$

Proof. We set $t_1 = t_0$ in (18). Problem (18), (19) (at $t_1 = t_0$) for $z(t, t_0, x, \tau, v(\cdot), \omega(\cdot))$ has the form of problem (11), (12). Formula (14) yields

$$z(t,t_0,x,\tau,v(\cdot),\omega(\cdot)) = \frac{1}{4\pi(t-t_0)}$$

$$\times\exp\left(-\frac{x_2^2+x_3^2}{4(t-t_0)}\right)\overset{x_2x_3}{*}(M(u_0(x)u_0(\tau))\overset{x_1}{*}F_{\xi_1}^{-1}[\varphi(v(\cdot)+i\xi_1^2\chi(t_0,t,\cdot),\omega(\cdot))](x_1))$$

$$-i\int_{t_0}^{t}\frac{1}{4\pi(t-s)}\exp\left(-\frac{x_2^2+x_3^2}{4(t-s)}\right)\overset{x_2x_3}{*}F_{\xi_1}^{-1}\left[F_{x_1}\left(\frac{\delta y(t_0,\tau,v(\cdot)+i\xi_1^2\chi(s,t,\cdot),\omega(\cdot))}{\delta\omega(s,x)}\right)(\xi_1)\right](x_1)ds.$$

Since z is symmetric about (t, x) and (t_1, τ),

$$z(t_0,t_1,\tau,x,v(\cdot),\omega(\cdot)) = \frac{1}{4\pi(t_1-t_0)}$$

$$\times\exp\left(-\frac{x_2^2+x_3^2}{4(t_1-t_0)}\right)\overset{x_2x_3}{*}(M(u_0(x)u_0(\tau))\overset{x_1}{*}F_{\xi_1}^{-1}[\varphi(v(\cdot)+i\xi_1^2\chi(t_0,t_1,\cdot),\omega(\cdot))](x_1))$$

$$-i\int_{t_0}^{t_1}\frac{1}{4\pi(t_1-s)}\exp\left(-\frac{x_2^2+x_3^2}{4(t_1-s)}\right)\overset{x_2x_3}{*}F_{\xi_1}^{-1}\left[F_{x_1}\left(\frac{\delta y(t_0,\tau,v(\cdot)+i\xi_1^2\chi(s,t_1,\cdot),\omega(\cdot))}{\delta\omega(s,x)}\right)(\xi_1)\right](x_1)ds.$$

Then

$$z(t_0,t_1,x,\tau,v(\cdot),\omega(\cdot)) = \frac{1}{4\pi(t_1-t_0)}$$

$$\times\exp\left(-\frac{\tau_2^2+\tau_3^2}{4(t_1-t_0)}\right)\overset{\tau_2\tau_3}{*}(M(u_0(x)u_0(\tau))\overset{\tau_1}{*}F_{\xi_1}^{-1}[\varphi(v(\cdot)+i\xi_1^2\chi(t_0,t_1,\cdot),\omega(\cdot))](\tau_1))$$

$$-i\int_{t_0}^{t_1}\frac{1}{4\pi(t_1-s)}\exp\left(-\frac{\tau_2^2+\tau_3^2}{4(t_1-s)}\right)\overset{\tau_2\tau_3}{*}F_{\xi_1}^{-1}\left[F_{\tau_1}\left(\frac{\delta y(t_0,x,v(\cdot)+i\xi_1^2\chi(s,t_1,\cdot),\omega(\cdot))}{\delta\omega(s,\tau)}\right)(\xi_1)\right](\tau_1)ds.$$

Using (12), we find the initial condition for Eq. (18):

$$z(t_0,t_1,x,\tau,v(\cdot),\omega(\cdot)) = \frac{1}{4\pi(t_1-t_0)}$$

$$\times\exp\left(-\frac{\tau_2^2+\tau_3^2}{4(t_1-t_0)}\right)\overset{\tau_2\tau_3}{*}(M(u_0(x)u_0(\tau))\overset{\tau_1}{*}F_{\xi_1}^{-1}[\varphi(v(\cdot)+i\xi_1^2\chi(t_0,t_1,\cdot),\omega(\cdot))](\tau_1))$$

$$-iM(u_0(x))\int_{t_0}^{t_1}\frac{1}{4\pi(t_1-s)}\exp\left(-\frac{\tau_2^2+\tau_3^2}{4(t_1-s)}\right)\overset{\tau_2\tau_3}{*}F_{\xi_1}^{-1}\left[F_{x_1}\left(\frac{\delta\varphi(v(\cdot)+i\xi_1^2\chi(s,t_1,\cdot),\omega(\cdot))}{\delta\omega(s,\tau)}\right)(\xi_1)\right](\tau_1)ds.$$

Equation (18) with this initial condition has the form of problem (8), (9), and formula (10) gives

$$z(t,t_1,x,\tau,v(\cdot),\omega(\cdot)) = \frac{1}{4\pi(t-t_0)}\exp\left(-\frac{x_2^2+x_3^2}{4(t-t_0)}\right)\delta(x_1)\overset{x}{*}F_{\xi_1}^{-1}\left[F_{x_1}\left[\frac{1}{4\pi(t_1-t_0)}\right.\right.$$

$$\left.\left.\times\exp\left(-\frac{\tau_2^2+\tau_3^2}{4(t_1-t_0)}\right)\overset{\tau_2\tau_3}{*}(M(u_0(x)u_0(\tau))\overset{\tau_1}{*}F_{\eta_1}^{-1}[\varphi(v(\cdot)+i\eta_1^2\chi(t_0,t_1,\cdot)+i\xi_1^2\chi(t_0,t,\cdot),\omega(\cdot))](\tau_1)\right](\xi_1)\right](x_1))$$

$$-i\frac{1}{4\pi(t-t_0)}\exp\Big(-\frac{x_2^2+x_3^2}{4(t-t_0)}\Big)\delta(x_1)\overset{x}{*}F_{\xi_1}^{-1}\Bigg[F_{x_1}\Bigg[M(u_0(x))\int_{t_0}^{t_1}\frac{1}{4\pi(t_1-s)}\exp\Big(-\frac{\tau_2^2+\tau_3^2}{4(t_1-s)}\Big)\overset{\tau_2\tau_3}{*}F_{\eta_1}^{-1}\Big[F_{x_1}\Big[\frac{\delta}{\delta\omega(s,\tau)}\varphi(v(\cdot)$$

$$+i\eta_1^2\chi(s,t_1,\cdot)+i\xi_1^2\chi(t_0,t,\cdot),\omega(\cdot))\Big](\eta_1)\Big](\tau_1)ds\Bigg](\xi_1)\Bigg](x_1)$$

$$-i\int_{t_0}^{t}\frac{1}{4\pi(t-s)}\exp\Big(-\frac{x_2^2+x_3^2}{4(t-s)}\Big)\delta(x_1)\overset{x}{*}F_{\xi_1}^{-1}\Big[F_{x_1}\Big[\frac{\delta}{\delta\omega(s,x)}y(t_1,\tau,v(\cdot)+i\xi_1^2\chi(s,t,\cdot),\omega(\cdot))\Big](\xi_1)\Big](x_1)ds$$

$$=\frac{1}{4\pi(t-t_0)}\exp\Big(-\frac{x_2^2+x_3^2}{4(t-t_0)}\Big)\overset{x_2x_3}{*}\Bigg\{\frac{1}{4\pi(t_1-t_0)}\exp\Big(-\frac{\tau_2^2+\tau_3^2}{4(t_1-t_0)}\Big)\overset{\tau_2\tau_3}{*}\{M(u_0(x)u_0(\tau))\overset{\tau_1}{*}F_{\xi_1}^{-1}[F_{\eta_1}^{-1}[\varphi(v(\cdot)$$

$$+i\eta_1^2\chi(t_0,t_1,\cdot)+i\xi_1^2\chi(t_0,t,\cdot),\omega(\cdot))](\tau_1)](x_1)\}\Bigg\}$$

$$-i\frac{1}{4\pi(t-t_0)}\exp\Big(-\frac{x_2^2+x_3^2}{4(t-t_0)}\Big)\overset{x_2x_3}{*}\Bigg\{M(u_0(x))\overset{x_1}{*}\int_{t_0}^{t_1}\frac{1}{4\pi(t_1-s)}\exp\Big(-\frac{\tau_2^2+\tau_3^2}{4(t_1-s)}\Big)\overset{\tau_2\tau_3}{*}F_{\xi_1}^{-1}\Big[F_{\eta_1}^{-1}\Big[F_{x_1}\Big[\frac{\delta}{\delta\omega(s,\tau)}\varphi(v(\cdot)$$

$$+i\eta_1^2\chi(s,t_1,\cdot)+i\xi_1^2\chi(t_0,t,\cdot),\omega(\cdot))\Big](\eta_1)\Big](\tau_1)\Big](x_1)ds\Bigg\}$$

$$-i\int_{t_0}^{t}\frac{1}{4\pi(t-s)}\exp\Big(-\frac{x_2^2+x_3^2}{4(t-s)}\Big)\overset{x_2x_3}{*}F_{\xi_1}^{-1}\Big[F_{x_1}\Big[\frac{\delta}{\delta\omega(s,x)}y(t_1,\tau,v(\cdot)+i\xi_1^2\chi(s,t,\cdot),\omega(\cdot))\Big](\xi_1)\Big](x_1)ds.$$

According to (14), we have

$$y(t_1,\tau,v(\cdot)+i\xi_1^2\chi(s,t,\cdot),\omega(\cdot))=\frac{1}{4\pi(t_1-t_0)}\exp\Big(-\frac{\tau_2^2+\tau_3^2}{4(t_1-t_0)}\Big)\overset{\tau_2\tau_3}{*}(M(u_0(\tau)))\overset{\tau_1}{*}F_{\eta_1}^{-1}[\varphi(v(\cdot)$$

$$+i\eta_1^2\chi(t_0,t_1,\cdot)+i\xi_1^2\chi(s,t,\cdot),\omega(\cdot))](\tau_1)-i\int_{t_0}^{t_1}\frac{1}{4\pi(t_1-\theta)}$$

$$\times\exp\Big(-\frac{\tau_2^2+\tau_3^2}{4(t_1-\theta)}\Big)\overset{\tau_2\tau_3}{*}F_{\eta_1}^{-1}\Big[F_{x_1}\Big[\frac{\delta\varphi(v(\cdot)+i\eta_1^2\chi(\theta,t_1,\cdot)+i\xi_1^2\chi(s,t,\cdot),\omega(\cdot))}{\delta\omega(\theta,\tau)}\Big](\eta_1)\Big](\tau_1)d\theta.$$

Substituting this expression and using the properties of Fourier transforms produces

$$z=\frac{1}{4\pi(t-t_0)}\exp\Big(-\frac{x_2^2+x_3^2}{4(t-t_0)}\Big)\overset{x_2x_3}{*}\Bigg\{\frac{1}{4\pi(t_1-t_0)}\exp\Big(-\frac{\tau_2^2+\tau_3^2}{4(t_1-t_0)}\Big)\overset{\tau_2\tau_3}{*}\{M(u_0(x)u_0(\tau))\overset{\tau_1}{*}F_{\xi_1}^{-1}[F_{\eta_1}^{-1}[\varphi(v(\cdot)$$

$$+i\eta_1^2\chi(t_0,t_1,\cdot)+i\xi_1^2\chi(t_0,t,\cdot),\omega(\cdot))](\tau_1)](x_1)\}\Bigg\}-i\frac{1}{4\pi(t-t_0)}\exp\Big(-\frac{x_2^2+x_3^2}{4(t-t_0)}\Big)\overset{x_2x_3}{*}\Bigg\{M(u_0(x))\overset{x_1}{*}\int_{t_0}^{t_1}\frac{1}{4\pi(t_1-s)}$$

$$\times \exp\left(-\frac{\tau_2^2+\tau_3^2}{4(t_1-s)}\right) \overset{\tau_2\tau_3}{*} F_{\xi_1}^{-1}\left[F_{\eta_1}^{-1}\left[F_{x_1}\left[\frac{\delta}{\delta\omega(s,\tau)}\varphi(v(\cdot)+i\eta_1^2\chi(s,t_1,\cdot)+i\xi_1^2\chi(t_0,t,\cdot),\omega(\cdot))\right](\eta_1)\right](\tau_1)\right](x_1)ds\right\}$$

$$-i\frac{1}{4\pi(t_1-t_0)}\exp\left(-\frac{\tau_2^2+\tau_3^2}{4(t_1-t_0)}\right) \overset{\tau_2\tau_3}{*} \left\{M(u_0(\tau)) \overset{\tau_1}{*} \int_{t_0}^{t}\frac{1}{4\pi(t-s)}\exp\left(-\frac{x_2^2+x_3^2}{4(t-s)}\right) \overset{x_2x_3}{*} F_{\eta_1}^{-1}\left[F_{\xi_1}^{-1}\left[F_{x_1}\left[\frac{\delta}{\delta\omega(s,x)}\varphi(v(\cdot)\right.\right.\right.\right.$$

$$\left.\left.\left.\left.+i\eta_1^2\chi(t_0,t_1,\cdot)+i\xi_1^2\chi(s,t,\cdot),\omega(\cdot)\right](\xi_1)\right](x_1)\right](\tau_1)ds\right\} - \int_{t_0}^{t}ds\int_{t_0}^{t_1}\frac{1}{4\pi(t_1-s)}$$

$$\times \exp\left(-\frac{x_2^2+x_3^2}{4(t-s)}\right) \overset{x_2x_3}{*} \left\{\frac{1}{4\pi(t_1-\theta)}\exp\left(-\frac{\tau_2^2+\tau_3^2}{4(t_1-\theta)}\right) \overset{\tau_2\tau_3}{*} F_{\xi_1}^{-1}\left[F_{x_1}\left[F_{\eta_1}^{-1}\left[F_{x_1}\right.\right.\right.\right.$$

$$\left.\left.\left.\left.\times\left[\frac{\delta^2\varphi(v(\cdot)+i\eta_1^2\chi(\theta,t_1,\cdot)+i\xi_1^2\chi(s,t,\cdot),\omega(\cdot))}{\delta\omega(s,x)\delta\omega(\theta,\tau)}\right](\eta_1)\right](\tau_1)\right](\xi_1)\right](x_1)\right\}d\theta.$$

In the third term from the end, the variables involved in the inverse Fourier transform are renamed so that η is switched to ξ and vice versa. As a result, we obtain (20). The theorem is proved.

6. SECOND MOMENT OF THE SOLUTION TO PROBLEM (1), (2)

Definition 3. The generalized second moment $M(u(t,x)u(t_1,\tau))$ of the solution to problem (1), (2) is defined as $z(t, t_1, x, \tau, 0, 0)$, where z is a (t, x)- and (t_1, τ)-symmetric generalized solution to problem (18), (19).

Theorem 9. *Let the conditions of Theorem* 6 *be satisfied. Then the generalized second moment of the solution to problem* (1), (2) *is given by the formula*

$$M(u(t,x)u(t_1,\tau)) = \frac{1}{4\pi(t-t_0)}\exp\left(-\frac{x_2^2+x_3^2}{4(t-t_0)}\right) \overset{x_2x_3}{*} \left\{\frac{1}{4\pi(t_1-t_0)}\right.$$

$$\left.\times\exp\left(-\frac{\tau_2^2+\tau_3^2}{4(t_1-t_0)}\right) \overset{\tau_2\tau_3}{*} \{M(u_0(x)u_0(\tau)) \overset{\tau_1}{*} F_{\xi_1}^{-1}[F_{\eta_1}^{-1}[\varphi(i\eta_1^2\chi(t_0,t_1,\cdot)+i\xi_1^2\chi(t_0,t,\cdot),0)](\tau_1)](x_1)\}\right\}$$

$$-i\frac{1}{4\pi(t-t_0)}\exp\left(-\frac{x_2^2+x_3^2}{4(t-t_0)}\right) \overset{x_2x_3}{*} \left\{M(u_0(x)) \overset{x_1}{*} \int_{t_0}^{t_1}\frac{1}{4\pi(t_1-s)}\exp\left(-\frac{\tau_2^2+\tau_3^2}{4(t_1-s)}\right) \overset{\tau_2\tau_3}{*} F_{\xi_1}^{-1}\left[F_{\eta_1}^{-1}\left[F_{x_1}\right.\right.\right.$$

$$\left.\left.\left.\times\left[\frac{\delta}{\delta\omega(s,\tau)}\varphi(i\eta_1^2\chi(s,t_1,\cdot)+i\xi_1^2\chi(t_0,t,\cdot),0)\right](\eta_1)\right](\tau_1)\right](x_1)ds\right\} - i\frac{1}{4\pi(t-t_0)} \tag{21}$$

$$\times\exp\left(-\frac{\tau_2^2+\tau_3^2}{4(t_1-t_0)}\right) \overset{\tau_2\tau_3}{*} \left\{M(u_0(\tau)) \overset{\tau_1}{*} \int_{t_0}^{t}\frac{1}{4\pi(t-s)}\exp\left(-\frac{x_2^2+x_3^2}{4(t-s)}\right) \overset{x_2x_3}{*} F_{\eta_1}^{-1}\left[F_{\xi_1}^{-1}\left[F_{x_1}\right.\right.\right.$$

$$\times\left[\frac{\delta}{\delta\omega(s,x)}\varphi(i\eta_1^2\chi(t_0,t_1,\cdot)+i\xi_1^2\chi(s,t,\cdot),0)\right](\xi_1)\Big](x_1)\Big](\tau_1)ds\Bigg\}-\int_{t_0}^{t}ds\int_{t_0}^{t_1}\frac{1}{4\pi(t-s)}$$

$$\times\exp\left(-\frac{x_2^2+x_3^2}{4(t-s)}\right)\overset{x_2x_3}{*}\left\{\frac{1}{4\pi(t_1-\theta)}\exp\left(-\frac{\tau_2^2+\tau_3^2}{4(t_1-\theta)}\right)\overset{\tau_2\tau_3}{*}F_{\xi_1}^{-1}\Big[F_{x_1}\Big[F_{\eta_1}^{-1}\Big[F_{x_1}\right.$$

$$\left.\times\left[\frac{\delta^2\varphi(i\eta_1^2\chi(\theta,t_1,\cdot)+i\xi_1^2\chi(s,t,\cdot),0)}{\delta\omega(s,x)\delta\omega(\theta,\tau)}\right](\eta_1)\Big](\tau_1)\Big](\xi_1)\Big](x_1)\right\}d\theta.$$

Proof. Using Definition 3 and setting $v = 0$ and $\omega = 0$ in (20), we obtain (21).

Theorem 10. *Suppose that the stochastic processes* u_0, ε, *and* f *in problem* (1), (2) *are independent; condition* (4) *is satisfied; the characteristic functional* φ_ε *of* ε *has a variational derivative* $\dfrac{\delta\varphi(v(\cdot)+i\eta_1^2\chi(\theta,t_1,\cdot)+i\xi_1^2\chi(s,t,\cdot))}{\delta v(s)}$ *in the neighborhood of the point* $v = 0$; *and* $M(u_0(x))$, $M(u_0(\tau))$, $M(u_0(x)u_0(\tau))$, *and* $M(f(t,x)f(t_1,\tau))$ *are locally summable functions. Then the generalized second moment of the solution to problem* (1), (2) *is given by*

$$M(u(t,x)u(t_1,\tau))=\frac{1}{4\pi(t-t_0)}\exp\left(-\frac{x_2^2+x_3^2}{4(t-t_0)}\right)\overset{x_2x_3}{*}\left\{\frac{1}{4\pi(t_1-t_0)}\right.$$

$$\left.\times\exp\left(-\frac{\tau_2^2+\tau_3^2}{4(t_1-t_0)}\right)\overset{\tau_2\tau_3}{*}\{M(u_0(x)u_0(\tau))\overset{\tau_1}{*}F_{\xi_1}^{-1}[F_{\eta_1}^{-1}[\varphi_\varepsilon(i\eta_1^2\chi(t_0,t_1,\cdot)+i\xi_1^2\chi(t_0,t,\cdot))](\tau_1)](x_1)\}\right\}$$

$$+\frac{1}{4\pi(t-t_0)}\exp\left(-\frac{x_2^2+x_3^2}{4(t-t_0)}\right)\overset{x_2x_3}{*}\left\{M(u_0(x))\overset{x_1}{*}\int_{t_0}^{t_1}\frac{1}{4\pi(t_1-s)}\exp\left(-\frac{\tau_2^2+\tau_3^2}{4(t_1-s)}\right)\overset{\tau_2\tau_3}{*}\{F_{\xi_1}^{-1}[F_{\eta_1}^{-1}\right.$$

$$\left.\times[\varphi_\varepsilon(i\eta_1^2\chi(s,t_1,\cdot)+i\xi_1^2\chi(t_0,t,\cdot))](\tau_1)](x_1)\overset{x_1}{*}M(f(s,\tau))\}ds\right\}+\frac{1}{4\pi(t_1-t_0)}$$

$$\times\exp\left(-\frac{\tau_2^2+\tau_3^2}{4(t_1-t_0)}\right)\overset{\tau_2\tau_3}{*}\left\{M(u_0(\tau))\overset{\tau_1}{*}\int_{t_0}^{t}\frac{1}{4\pi(t-s)}\exp\left(-\frac{x_2^2+x_3^2}{4(t-s)}\right)\overset{x_2x_3}{*}\{F_{\eta_1}^{-1}[F_{\xi_1}^{-1}\right. \tag{22}$$

$$\left.\times[\varphi_\varepsilon(i\eta_1^2\chi(t_0,t_1,\cdot)+i\xi_1^2\chi(s,t,\cdot))](x_1)](\tau_1)\overset{x_1}{*}M(f(s,x))\}ds\right\}+\int_{t_0}^{t}ds\int_{t_0}^{t_1}\frac{1}{4\pi(t-s)}$$

$$\times\exp\left(-\frac{x_2^2+x_3^2}{4(t-s)}\right)\overset{x_2x_3}{*}\left\{\frac{1}{4\pi(t_1-\theta)}\exp\left(-\frac{\tau_2^2+\tau_3^2}{4(t_1-\theta)}\right)\overset{\tau_2\tau_3}{*}\{F_{\xi_1}^{-1}[F_{\eta_1}^{-1}\right.$$

$$\left.\times[\varphi_\varepsilon(i\eta_1^2\chi(\theta,t_1,\cdot)+i\xi_1^2\chi(s,t,\cdot))](\tau_1)](x_1)\overset{\tau_1}{*}M(f(s,x)f(\theta,\tau))\}\right\}d\theta.$$

Proof. Since ε and f are independent, $\varphi(v(\cdot), \omega(\cdot)) = \varphi_\varepsilon(v(\cdot))\varphi_f(\omega(\cdot))$, where $\varphi_\varepsilon(v(\cdot))$ and $\varphi_f(\omega(\cdot))$ are the respective characteristic functionals of ε and f. Furthermore,

$$\frac{\delta\varphi_f(0)}{\delta\omega(s,\tau)} = iMf(s,\tau), \quad \varphi_f(0) = 1, \text{ and } \frac{\delta^2\varphi_f(0)}{\delta\omega(t,x)\delta\omega(t_1,\tau)} = -M(f(t,x)f(t_1,\tau)).$$

Using these relations and the properties of Fourier transforms, we obtain

$$-i\frac{1}{4\pi(t-t_0)}\exp\Big(-\frac{x_2^2+x_3^2}{4(t-t_0)}\Big)\overset{x_2x_3}{*}\Big\{M(u_0(x))\overset{x_1}{*}\int_{t_0}^{t_1}\frac{1}{4\pi(t_1-s)}\exp\Big(-\frac{\tau_2^2+\tau_3^2}{4(t_1-s)}\Big)\overset{\tau_2\tau_3}{*}F_{\xi_1}^{-1}\Big[F_{\eta_1}^{-1}\Big[F_{x_1}$$

$$\times\Big[\varphi_\varepsilon(i\eta_1^2\chi(s,t_1,\cdot)+i\xi_1^2\chi(t_0,t,\cdot))\frac{\delta}{\delta\omega(s,\tau)}\varphi_f(0)\Big](\eta_1)\Big](\tau_1)\Big](x_1)ds\Big\}$$

$$= \frac{1}{4\pi(t-t_0)}\exp\Big(-\frac{x_2^2+x_3^2}{4(t-t_0)}\Big)\overset{x_2x_3}{*}\Big\{M(u_0(x))\overset{x_1}{*}\int_{t_0}^{t_1}\frac{1}{4\pi(t_1-s)}\exp\Big(-\frac{\tau_2^2+\tau_3^2}{4(t_1-s)}\Big)\overset{\tau_2\tau_3}{*}F_{\xi_1}^{-1}[F_{\eta_1}^{-1}$$

$$\times[\varphi_\varepsilon(i\eta_1^2\chi(s,t_1,\cdot)+i\xi_1^2\chi(t_0,t,\cdot))F_{x_1}[M(f(s,\tau))](\eta_1)](\tau_1)](x_1)ds\Big\}$$

$$= \frac{1}{4\pi(t-t_0)}\exp\Big(-\frac{x_2^2+x_3^2}{4(t-t_0)}\Big)\overset{x_2x_3}{*}\Big\{M(u_0(x))\overset{x_1}{*}\int_{t_0}^{t_1}\frac{1}{4\pi(t_1-s)}\exp\Big(-\frac{\tau_2^2+\tau_3^2}{4(t_1-s)}\Big)\overset{\tau_2\tau_3}{*}(F_{\xi_1}^{-1}[F_{\eta_1}^{-1}$$

$$\times[\varphi_\varepsilon(i\eta_1^2\chi(s,t_1,\cdot)+i\xi_1^2\chi(t_0,t,\cdot))(\tau_1)](x_1)\overset{x_1}{*}M(f(s,\tau)))ds\Big\};$$

$$-i\frac{1}{4\pi(t_1-t_0)}\exp\Big(-\frac{x_2^2+x_3^2}{4(t_1-t_0)}\Big)\overset{\tau_2\tau_3}{*}\Big\{M(u_0(\tau))\overset{\tau_1}{*}\int_{t_0}^{t}\frac{1}{4\pi(t-s)}\exp\Big(-\frac{x_2^2+x_3^2}{4(t-s)}\Big)\overset{x_2x_3}{*}F_{\eta_1}^{-1}\Big[F_{\xi_1}^{-1}\Big[F_{x_1}$$

$$\times\Big[\varphi_\varepsilon(i\eta_1^2\chi(t_0,t_1,\cdot)+i\xi_1^2\chi(s,t,\cdot))\frac{\delta}{\delta\omega(s,x)}\varphi_f(0)\Big](\xi_1)\Big](x_1)\Big](\tau_1)ds\Big\}$$

$$= \frac{1}{4\pi(t_1-t_0)}\exp\Big(-\frac{\tau_2^2+\tau_3^2}{4(t_1-t_0)}\Big)\overset{\tau_2\tau_3}{*}\Big\{M(u_0(\tau))\overset{\tau_1}{*}\int_{t_0}^{t}\frac{1}{4\pi(t-s)}\exp\Big(-\frac{x_2^2+x_3^2}{4(t-s)}\Big)\overset{x_2x_3}{*}F_{\eta_1}^{-1}[F_{\xi_1}^{-1}$$

$$\times[\varphi_\varepsilon(i\eta_1^2\chi(t_0,t_1,\cdot)+i\xi_1^2\chi(s,t,\cdot))F_{x_1}[M(f(s,x))](\xi_1)](x_1)](\tau_1)ds\Big\}$$

$$= \frac{1}{4\pi(t_1-t_0)}\exp\Big(-\frac{\tau_2^2+\tau_3^2}{4(t_1-t_0)}\Big)\overset{\tau_2\tau_3}{*}\Big\{M(u_0(\tau))\overset{\tau_1}{*}\int_{t_0}^{t}\frac{1}{4\pi(t-s)}\exp\Big(-\frac{x_2^2+x_3^2}{4(t-s)}\Big)\overset{x_2x_3}{*}\{F_{\eta_1}^{-1}[F_{\xi_1}^{-1}$$

$$\times[\varphi_\varepsilon(i\eta_1^2\chi(t_0,t_1,\cdot)+i\xi_1^2\chi(s,t,\cdot))](x_1)](\tau_1)\overset{x_1}{*}M(f(s,x))\}ds\Bigg\};$$

$$-\int_{t_0}^{t}ds\int_{t_0}^{t_1}\frac{1}{4\pi(t-s)}\exp\Big(-\frac{x_2^2+x_3^2}{4(t-s)}\Big)\overset{x_2x_3}{*}\Bigg\{\frac{1}{4\pi(t_1-\theta)}\exp\Big(-\frac{\tau_2^2+\tau_3^2}{4(t_1-\theta)}\Big)\overset{\tau_2\tau_3}{*}F_{\xi_1}^{-1}\Big[F_{x_1}\Big[F_{\eta_1}^{-1}\Big[F_{x_1}$$

$$\times\Big[\varphi_\varepsilon(i\eta_1^2\chi(\theta,t_1,\cdot)+i\xi_1^2\chi(s,t,\cdot))\frac{\delta^2\varphi_f(0)}{\delta\omega(s,x)\delta\omega(\theta,\tau)}\Big](\eta_1)\Big](\tau_1)\Big](\xi_1)\Big](x_1)\Bigg\}d\theta$$

$$=\int_{t_0}^{t}ds\int_{t_0}^{t_1}\frac{1}{4\pi(t-s)}\exp\Big(-\frac{x_2^2+x_3^2}{4(t-s)}\Big)\overset{x_2x_3}{*}\Bigg\{\frac{1}{4\pi(t_1-\theta)}\exp\Big(-\frac{\tau_2^2+\tau_3^2}{4(t_1-\theta)}\Big)\overset{\tau_2\tau_3}{*}F_{\xi_1}^{-1}[F_{\eta_1}^{-1}$$

$$\times[\varphi_\varepsilon(i\eta_1^2\chi(\theta,t_1,\cdot)+i\xi_1^2\chi(s,t,\cdot))F_{x_1}[F_{\tau_1}[M(f(s,x)f(\theta,\tau))](\eta_1)](\tau_1)](\xi_1)](x_1)\Bigg\}d\theta$$

$$=\int_{t_0}^{t}ds\int_{t_0}^{t_1}\frac{1}{4\pi(t-s)}\exp\Big(-\frac{x_2^2+x_3^2}{4(t-s)}\Big)\overset{x_2x_3}{*}\Bigg\{\frac{1}{4\pi(t_1-\theta)}\exp\Big(-\frac{\tau_2^2+\tau_3^2}{4(t_1-\theta)}\Big)\overset{\tau_2\tau_3}{*}\{F_{\xi_1}^{-1}[F_{\eta_1}^{-1}$$

$$\times[\varphi_\varepsilon(i\eta_1^2\chi(\theta,t_1,\cdot)+i\xi_1^2\chi(s,t,\cdot))](\tau_1)](x_1)\overset{\tau_1}{*}M(f(s,x)f(\theta,\tau))\}\Bigg\}d\theta.$$

Substituting these relations into (21) yields (22). The theorem is proved.

7. MIXED SECOND MOMENT OF THE SOLUTION TO PROBLEM (1), (2)

Applying variational differentiation to the formula derived for $y(t, x, v(\cdot), \omega(\cdot))$, we can find mixed moments, for example, $M(u(t,x)\varepsilon(\tau)) = -i\dfrac{\delta y(t,x,0,0)}{\delta v(\tau)}$.

Theorem 11. *Suppose that* $\varepsilon(t)$ *and* $f(t, x)$ *are independent stochastic processes, the characteristic functional* $\varphi_\varepsilon(v(\cdot))$ *of* $\varepsilon(t)$ *has* $v(\cdot)$*-continuous variational derivatives up to the second order inclusive,* $M(u_0(x))$ *is summable on* $\mathbb{R}^3$, *and* $M(f(t, x))$ *is summable on* $T\times\mathbb{R}^3$. *Then*

$$M(u(t,x)\varepsilon(s)) = -i\frac{1}{4\pi(t-t_0)}\exp\Big(-\frac{x_2^2+x_3^2}{4(t-t_0)}\Big)\overset{x_2x_3}{*}\Big(M(u_0(x))\overset{x_1}{*}F_{\xi_1}^{-1}\Big[\frac{\delta\varphi_\varepsilon(i\xi_1^2\chi(t_0,t,\cdot))}{\delta v(\tau)}\Big](x_1)\Big)$$

$$-i\int_{t_0}^{t}\frac{1}{4\pi(t-s)}\exp\Big(-\frac{x_2^2+x_3^2}{4(t-s)}\Big)\overset{x_2x_3}{*}\Bigg\{F_{\xi_1}^{-1}\Big[F_{x_1}\Big[\frac{\delta\varphi_\varepsilon(i\xi_1^2\chi(s,t,\cdot))}{\delta v(\tau)}\Big](\xi_1)\Big](x_1)\overset{x_1}{*}M(f(s,x))\Bigg\}ds.$$

ACKNOWLEDGMENTS

The authors are grateful to Professor A.V. Fursikov for helpful discussion of these results.

REFERENCES

1. A. V. Fursikov, "Closure of Chain of Moment Equations at High Reynolds Numbers," in *Nonclassical Equations and Mixed-Type Equations* (Inst. Mat. Sib. Otd. Akad. Nauk SSSR, Novosibirsk, 1990), pp. 231–250 [in Russian].
2. A. D. Venttsel' and M. N. Freindlin, *Fluctuations in Dynamical Systems Subject to Small Random Perturbations* (Nauka, Moscow, 1979) [in Russian].
3. G. Adomian, *Stochastic Systems* (Academic, New York, 1983; Mir, Moscow, 1987).
4. V. I. Klyatskin, *Stochastic Equations and Waves in Randomly Inhomogeneous Media* (Nauka, Moscow, 1980) [in Russian].
5. V. G. Zadorozhnii, *Methods of Variational Analysis* (NITs RKhD, Moscow, 2006) [in Russian].
6. A. N. Kolmogorov and S. V. Fomin, *Elements of the Theory of Functions and Functional Analysis* (Graylock, Rochester, N.Y., 1957/1961; Nauka, Moscow, 1972).
7. V. S. Vladimirov, *Generalized Functions in Mathematical Physics* (Nauka, Moscow, 1976; Mir, Moscow, 1979).